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Renorm-group symmetry for functionals of boundary value problem solutions

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Abstract

Recent advances in generalizing the renorm-group algorithm for boundary value problems of mathematical physics and the related concept of the renorm-group symmetry, previously formulated with reference to models based on differential equations, are revisited. The algorithm and symmetry are now formulated for models with nonlocal (integral) equations. Examples illustrate how the updated algorithm applies to models with nonlocal terms appearing as linear functionals of the solution.

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1. Introduction

The renorm-group symmetry (RGS) was introduced in mathematical physics in the beginning of the 1990s as a result of combining the QFT Stueckelberg–Bogoliubov [1–3] renormalization group (RG)[4] generalized in a form of functional self-similarity [5] with the Sophus Lie group formalism. Here, the central idea is tightly connected with Bogoliubov's RG method [6] of improving an approximate solution of the QFT problem in the vicinity of a solution singularity.

The first successful attempt [7] concerned applying RG ideas to a problem of generating higher harmonics in a plasma. This problem, after some simplification, reduced to a pair of partial differential equations (PDEs) with the boundary parameter (solution 'characteristic') explicitly included. It was proved that these DEs admit an exact symmetry group then used to construct the desired nonlinear solution of the boundary value problem (BVP) for nonvanishing values of the boundary parameter.

[4] The symmetry underlying the 'QFT RG' is an *exact symmetry of a solution* in contrast to some other constructions, such as the 'Wilson renormalization group'.

The RG concept was transferred to mathematical physics with the same pragmatic goal in mind: 'improving' the solution behaviour in the vicinity of a singularity. For the BVPs based on DEs, we developed the RG algorithm (see, e.g., [7, 8] and the review [9]) that unites the RG ideology of QFT with a regular symmetry construction procedure for BVP solutions. Because of this algorithm, there also arose the concept of the RGS for BVP solutions: these symmetries result from a calculation procedure similar to that used in modern group analysis.

Initially [7], applying the RG algorithm was mainly limited to problems based on DEs, although this algorithm can be used formally in any problem for which a regular way of calculating symmetries for the basic equations can be specified. Hence, transition to such objects, which until recently were not a subject of group analysis, in particular, to integral and integro-differential equations (IDEs), essentially expands the area of RGS applications.

In problems with involved equations, e.g., in transfer theory with integro-differential Boltzmann equation or in QFT with an infinite chain of coupled integro-differential Dyson–Schwinger equations, only some solution components or their integrated characteristics satisfy a sufficiently simple symmetry. Thus, in the one-velocity plane transfer problem, the RGS property is related [5] to the asymptotics of the 'density of particles, moving deep into the medium' $n_+(x), x \to \infty$, not entering the Boltzmann equation[5]. In such problems, integral relations form the problem skeleton. But they can appear as some independent objects for applying the RGS constructed for solutions of DEs. Frequently, not the solution itself in its entire range of the variables and parameters but rather some integral characteristic, a solution functional, is of physical interest. This characteristic can appear, for example, as a result of averaging (integrating) over one of the independent variables or of transition to a new integral representation, for example, a Fourier representation.

This report is structured as follows. In section 2, one finds an introductory example of the RGS algorithm in mathematical physics, illustrated by a solution of a simple BVP. In section 3, a generalization of the RG algorithm, developed earlier for BVPs with DEs [8], is reformulated for models with nonlocal terms, including integral equations and IDEs. Section 4 contains two examples of application of new algorithm. In the conclusion, we list some recent results obtained by this modified RG algorithm and discuss feasible prospects.

2. Illustrative examples of the RGS algorithm

2.1. Notation, terminology and definitions

Generally, the RG can be defined as a continuous one-parameter group of specific transformations of a partial solution (or solution characteristic) of a problem, a solution that is fixed by boundary condition (BC). The RG transformation involves BC parameters and corresponds to some change in how this condition is imposed.

For a given solution of some physical problem, the RG transformation is defined in the simplest case as *simultaneous one-parameter group transformations R_t of two variables*, ξ and g; for example,

$$R_t : \{\xi \to \xi' = \xi t, g \to g' = \bar{g}(t, g)\}, \tag{1}$$

the first being the scaling of a coordinate ξ (or reference point) and the second—a more complicated functional transformation of a solution characteristic g. Hence, the RG transformation corresponds to a change in the parameterization for the *same* solution, and the function $\bar{g}$ satisfies the equation

$$\bar{g}(\xi t, g) = \bar{g}(\xi, \bar{g}(t, g)), \qquad \bar{g}(1, g) = g, \tag{2}$$

[5] This is representable as the integral $\int_0^1 n(x, \vartheta)\, \mathrm{d}\cos\vartheta$ of the kinetic equation solution $n(x, \vartheta)$.

which guarantees fulfilment of the group property $R_{\tau t} = R_\tau R_t$. These are just the RG functional equations and transformations for a massless QFT model with one coupling [3]. In that case, $\xi = Q^2/\mu^2$ is the ratio of a four-momentum Q squared to a 'normalization' momentum μ squared, and g is a coupling constant, while $\bar{g}$ is the so-called effective coupling.

In general, we can consider transformations T_a,

$$\bar{x}^i = f^i(x, a), \qquad f^i(x, a_0) = x^i, \qquad i = 1, \ldots, n, \tag{3}$$

depending on a real parameter a, where $x \in \mathbb{R}^n$. A set G of these transformations forms a one-parameter local group if the functions $f^i(x, a)$ satisfy the composition rule $T_b T_a = T_{\phi(a,b)}$,

$$f^i(f^i(x, a), b) = f^i(x, \phi(a, b)), \qquad \phi(a, a_0) = a, \phi(a_0, b) = b, \tag{4}$$

where $\phi(a, b)$ is a three times continuously differentiable function of a and b. But any composition law can be transformed [10, chapter 7] to the simplest form with $\phi(a, b) = a + b$ and $a_0 = 0$ in (3) that is equivalent to additive version of equations (1) and (2) with $a = \ln t$; $b = \ln \tau$ and $T_{\ln t} = R_t$.

Geometrically, transformations (3) mean that any point $x \in \mathbb{R}^n$ is carried by these transformations into the point $\bar{x}$ whose locus is a continuous curve passing through x (called a path curve of the group G). Group property (4) means that any point of a path curve is carried by G into points of the same curve. The locus of the images $T_a(x)$ is also called the G-orbit of the point x. The correspondence between transformations (3) for $\phi(a, b) = ab$ and (1) means that for RG transformations, a curve in the plane $\{x, g\}$ representing the solution of a physical problem is the path curve of the RG R_t. In other words, the solution of the problem coincides with the R_t-orbit of a boundary manifold, the point $\{x = x_0, g = g_0\}$. Under the RG transformations, the reference (boundary) point $\{x_0, g_0\}$ is shifted to some other value $\{x_1, g_1\}$, while the solution remains unaltered, i.e., the solution curve is the invariant manifold of the group R_t (like the invariant charge in QFT [4]).

Hence, the general problem of seeking the RG transformations can be reformulated as follows: the solution of the physical problem should coincide with the orbit of the RG.

In mathematical physics, a solution of a physical problem usually appears as a solution of some BVP. Then, the corresponding RG transformation can be obtained from the symmetry group related to this BVP with the BC condition also involved in the group transformation. The key point here is that the relevant symmetry group is calculated by a regular procedure of modern group analysis, provided the problem is formulated in terms of DEs (or IDEs).

Let a model be given by a system of kth-order DEs, identified with its frame,

$$F_\sigma(x, u, u_{(1)}, \ldots, u_{(k)}) = 0, \qquad \sigma = 1, \ldots, s. \tag{5}$$

In this paper, we use the terminology of differential algebra and notation for variables accepted in group analysis [11]:

$$x = \{x^i\}, \qquad u = \{u^\alpha\}, \qquad u_{(1)} = \left\{u_i^\alpha\right\}, \qquad u_{(2)} = \left\{u_{ij}^\alpha\right\}, \ldots, \tag{6}$$

where $\alpha = 1, \ldots, m$ and $i, j, \ldots = 1, \ldots, n$. The variables x and u are respectively called independent variables and *differential variables*, having the consecutive derivatives $u_{(1)}, u_{(2)}, \ldots$. Differential variables are related by a system of equations

$$u_i^\alpha = D_i(u^\alpha), \qquad u_{ij}^\alpha = D_j\left(u_i^\alpha\right) = D_j D_i(u^\alpha), \ldots \tag{7}$$

via the total differentiation operator

$$D_i = \frac{\partial}{\partial z^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots. \tag{8}$$

A locally analytic function of variables (6), $F(x, u, u_{(1)}, \ldots, u_{(k)})$ for example, with the highest-order derivative k is called a differential function of the kth order, and a set of all such functions with any values of k forms the space of differential functions $\mathcal{A}[x, u]$. Any function $F \in \mathcal{A}[x, u]$ yields a differential manifold $[F]$, determined by an infinite system of equations

$$[F]: \quad F = 0, \qquad D_i F = 0, \qquad D_i D_j F = 0, \ldots. \tag{9}$$

The manifold $[F]$ is called the frame of the kth-order PDE

$$F\left(x, u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^k u}{\partial x^k}\right) = 0. \tag{10}$$

By *definition*, a system of sth-order DEs is said to be invariant under a group G if the frame of the system is an invariant manifold for the extension of the group G to the sth-order derivatives [10, p 209]. With an infinitesimal group generator

$$X = \xi^i \partial_{x^i} + \eta^\alpha \partial_{u^\alpha}, \qquad \xi^i, \eta^\alpha \in \mathcal{A}, \tag{11}$$

with coordinates ξ^i and η^α, which are functions of the group variables $\{x^i, u^\alpha\}$, this definition leads to an invariance criterion of the form

$$X_{(k)} F_{\sigma\,|[F_\sigma]} = 0, \qquad \sigma = 1, \ldots, s, \tag{12}$$

where $X_{(k)}$ denotes X extended to all the derivatives involved in F_σ and the symbol $|_{[F]}$ means evaluated on frame (9). Solving a system of linear homogeneous PDEs (called the *determining equations*) for the coordinates ξ^i and η^α gives a set of infinitesimal operators (11) (or group generators) corresponding to the admitted vector field of the symmetry group G and forming a Lie algebra L.

2.2. RGS construction: an idea and its simple realization

Let the Lie group G with generator

$$X = \xi^t \partial_t + \xi^x \partial_x + \eta \partial_y \tag{13}$$

be defined for the system of the first-order PDEs

$$y_t = F(t, x, y, y_x). \tag{14}$$

The typical BVP for (14) is the Cauchy problem with boundary manifold defined by

$$t = 0, \qquad y = \psi(x). \tag{15}$$

Solution of this Cauchy problem is the G-invariant solution iff for any generator (13), function ψ satisfies the equation [12, section 29]

$$\eta(0, x, \psi) - \xi^x(0, x, \psi)\partial_x \psi - \xi^t(0, x, \psi) F(0, x, \psi, \partial_x \psi) = 0. \tag{16}$$

The solution of Cauchy problem (14), (15) coincides with the orbit of the group G, and the boundary manifold is *not* the invariant manifold of the group.

This example gives an instructive idea for constructing generators of RGSs. The milestones here are (a) considering the BVP in the extended space of group variables that involve parameters of BCs in group transformations, (b) calculating the admitted group using the infinitesimal approach, (c) checking the invariance condition akin to (16) to find the symmetry group with the orbit that coincides with the BVP solution, and (d) using the RGS to find the improved (renormalized) solution of the BVP.

The whole algorithm was described in detail in our previous publications [8, 9]; here we only give a general grasp of the problem using a trivial example, the BVP for the Hopf equation

$$v_z + v v_x = 0, \qquad v(0, x) = \epsilon U(x), \tag{17}$$

where U is an invertible function of x. Introducing $u = \epsilon v$, we insert the boundary amplitude directly in the input equation:

$$u_z + \epsilon u u_x = 0, \qquad u(0, x) = U(x). \tag{18}$$

For small values of $\epsilon z \ll 1$, i.e., near the boundary, $z \to 0$, or for a small amplitude at the boundary, $\epsilon \to 0$, a perturbation theory solution of (18) has the form of a truncated power series in ϵz,

$$u = U - \epsilon z U U_x + O((\epsilon z)^2). \tag{19}$$

It is obvious that this solution is invalid for large distances from the boundary, when $\epsilon z U_x \backsimeq 1$. The RGS gives a way to improve the perturbation theory result and restore the correct structure of the BVP solution in the vicinity of a singularity (in the event that such singularity appears for some finite value of z).

With the goal of obtaining this symmetry, we extend the list of variables involved in the group transformations, adding the parameter ϵ to the list of independent variables. We then calculate the admitted symmetry group G with the generator

$$X = \xi^z \partial_z + \xi^x \partial_x + \xi^\epsilon \partial_\epsilon + \eta \partial_u \tag{20}$$

using the classical Lie calculational algorithm (see, e.g., [10]) with infinitesimal criterion (12). Solving the determining equations gives the coordinates of generator (20),

$$\xi^z = \psi^1, \qquad \xi^x = \epsilon u \psi^1 + \psi^2 + x(\psi^3 + \psi^4), \qquad \xi^\epsilon = \epsilon \psi^4, \qquad \eta = u\psi^3, \tag{21}$$

where $\psi^i, i = 2, 3, 4$, are arbitrary functions of ϵ, u, and $x - \epsilon u z$ and ψ^1 being an arbitrary function of all the group variables. These formulae define an infinite-dimensional Lie algebra with four generators (in the case where amplitude ϵ is not involved in the transformation, we have only three generators; see, e.g., [11, p 222])

$$\begin{aligned} &X_1 = \psi^1(\partial_z + \epsilon u \partial_x), \qquad && X_2 = \psi^2 \partial_x, \\ &X_3 = \psi^3(x\partial_x + u\partial_u), \qquad && X_4 = \psi^4(\epsilon\partial_\epsilon + x\partial_x). \end{aligned} \tag{22}$$

Suppose that a particular solution of BVP (18), $u - W(z, x, \epsilon) = 0$, which defines an invariant manifold of group (20), (21) is known. The corresponding invariance condition evaluated on frame (18) is similar to (12):

$$(W - xW_x)\psi^3 - W_x\psi^2 - (\epsilon W_\epsilon + xW_x)\psi^4 = 0. \tag{23}$$

This equation is valid for all z. Hence, it remains valid for $z = 0$, when W is replaced with $U(x)$. In this limit, $z \to 0$, condition (23) gives a relation between the $\psi^i, i = 2, 3, 4$ (no restrictions are imposed on ψ^1), that can be easily prolonged on $z \neq 0$,

$$\psi^2 = -\chi(\psi^3 + \psi^4) + (u/U_\chi)\psi^3, \qquad \chi = x - \epsilon u z, \tag{24}$$

where the derivative U_χ should be expressed, due to BC, either in terms of χ or u. By substituting (24) in (21), we obtain a group of a smaller dimension with generators

$$R_1 = \psi^1(\partial_z + \epsilon u \partial_x), \qquad R_2 = u\psi^3[(\epsilon z + 1/U_\chi)\partial_x + \partial_u], \qquad R_3 = \epsilon\psi^4(zu\partial_x + \partial_\epsilon). \tag{25}$$

The above procedure, which transforms (22) to (25), is the *restriction of the group* (20) *on a particular solution*.

The BVP solution defines a manifold, that, by construction, turns to be invariant for any generator R_i. Hence, (25) defines the desired RGSs. This means that the BVP solution can be constructed by use any of generators in (25), the generator R_3 for example. Without loss of generality, we choose $\epsilon\psi^4 = 1$ and obtain the finite RG transformations (a is a group parameter)

$$x' = x + azu, \qquad \epsilon' = \epsilon + a, \qquad z' = z, \qquad u' = u, \tag{26}$$

where z and u are invariants of the RG transformations while the transformations of ϵ and x are translations, which also depend on z and u in the case of x. For $\epsilon = 0$, in view of (19), we have $x = H(u)$, where $H(u)$ is a function inverse to $U(x)$. Eliminating a, z, u from (26) and omitting the primes on variables, we obtain the desired solution of BVP (18) in the implicit form

$$x - \epsilon zu = H(u). \tag{27}$$

This in fact is the improved perturbation theory solution (19), which is valid not only for small $\epsilon z \ll 1$, provided dependence (27) can be resolved uniquely. Depending upon $H(u)$ it gives either proper singular behaviour at some finite $z \to z_{\text{sing}}$ or correct asymptotic behaviour at $z \to \infty$.

The peculiarity of the procedure for constructing RGSs is the multi-choice first step, which depends on how the boundary conditions are formulated and the form in which the admitted symmetry group is calculated. For example, instead of calculating the Lie point symmetry group, we can consider the Lie–Bäcklund symmetries [15] with the canonical generator $R = \kappa\partial_u$, where κ depends not only on z, x, ϵ and u but also on higher-order derivatives of u. We can seek κ in the form of a power series in ϵ, and invariance condition (23) is formulated as vanishing of κ at $z = 0$. Depending on the choice of the zeroth-order term representation, we obtain either an infinite or a truncated power series for κ, for example, a form linear in ϵ,

$$R = \kappa\partial_u, \qquad \kappa = 1 - \frac{u_x}{U_x(u)} - \epsilon z u_x. \tag{28}$$

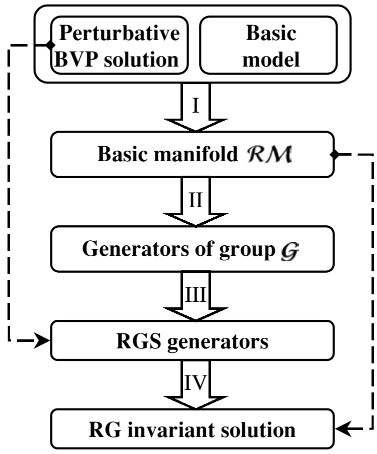
This RG generator (28) is equivalent to the Lie point generator R_2 in (25) and therefore gives the same result.

Another possibility for calculating RGSs for BVP (18) is offered by taking some additional differential constraints consistent with BCs and input equations into account. For example, if the boundary condition in (18) is linear in its argument, $U(x) = -x$, the differential constraint can be chosen as $u_{xx} = 0$; this equality reflects the invariance of the original equation with respect to the second-order Lie–Bäcklund symmetry group. Calculating the Lie point symmetry group for the joint system of this constraint and the Hopf equation gives another way to find RGSs for BVP (18).

The above example demonstrates the key features of the RGS method in mathematical physics. The details of the general approach are discussed in the following section.

3. The scheme of the RG algorithm for nonlocal problems

This formulation preserves the former general construction scheme of the RG algorithm (shown in the figure) as four consecutive steps [8, 9]:



(I) constructing the basic *manifold*,
(II) calculating the admitted *symmetry group* $\mathcal{G}$,
(III) *restricting* it on the particular BVP solution and constructing $\mathcal{RG}$, and
(IV) seeking an *analytic solution*.

But how these steps are realized varies significantly, which is most vividly shown in the first two steps of the algorithm [13, 14], related to constructing the nonlocal basic manifold and calculating the admitted symmetry group. Here, in view of the absence of a regular computational algorithm (similar to the Lie algorithm for DEs), various realizations are possible in performing step II. As an illustration, we choose and elaborate the variant based on using a canonical operator [13, 14, 18, 19]. Having in mind a reduced description of the solution in terms of the integrated characteristic, the solution functional, we describe the procedure for prolonging the RG operator on nonlocal variables [13, 14]. It is essential that knowing a solution in an explicit form is not required in this case.

3.1. Constructing the RG manifold

The initial issue is to construct the RGS and appropriate transformations that involve the parameters of partial solution. Therefore, the purpose of step I is to include all the parameters, both from the equations and from the BCs on which a particular solution depends, in group transformations in one or another way. This purpose is achieved by constructing a special manifold $\mathcal{RM}$ given by a system that consists of s kth-order DEs (5) and q nonlocal relations

$$F_\sigma(z, u, u_{(1)}, \ldots, u_{(r)}, J(u)) = 0, \qquad \sigma = 1 + s, \ldots, q + s. \tag{29}$$

The nonlocal variables $J(u)$ here are introduced by integrations,

$$J(u) = \int \mathcal{F}(u(z))\,\mathrm{d}z. \tag{30}$$

The presence of relations (29) in the system determining $\mathcal{RM}$ characterizes the basic difference between the case of a nonlocal problem and the case of a BVP for DEs, for which $\mathcal{RM}$ is a differential manifold.

3.2. Calculating the transformation group $\mathcal{G}$

Step II is to calculate the widest admitted symmetry group $\mathcal{G}$ for system (5), (29). An essential change of the RG algorithm is required here compared with its realization for a differential manifold $\mathcal{RM}$. Indeed, in application to an $\mathcal{RM}$ defined only by DE system (5), the question is about a local group of transformations in a space of differential functions $\mathcal{A}$, for which system (5) remains unchanged.

Meanwhile, the classical Lie algorithm using the infinitesimal approach seems to be inapplicable to a manifold $\mathcal{RM}$ set by system (5), (29). The issue is that the $\mathcal{RM}$ in this case is not determined *locally* in the space of differential functions. Therefore, the main advantage of the Lie computational algorithm, namely, representation of the determining equations as an over-determined system of equations is not realized here. Furthermore, the procedure for prolongation of the group operator of point transformations on nonlocal variables is not defined in the framework of classical group analysis.

In modifying the RG algorithm, we rely on the direct method for calculating symmetries advanced in [16, 17] for finding symmetries for Boltzmann kinetic equation, the equations of motion of viscous-elastic media, and the Vlasov–Maxwell equations in the kinetic theory of plasma. This method is based on a generalization of the symmetry group, the so-called Lie–Bäcklund symmetry group (the terms 'higher' or 'generalized' symmetry are also used), defined by the generator of form (11) prolonged on all higher-order derivatives,

$$
\begin{aligned}
&X = \xi^i \partial_{z^i} + \eta^\alpha \partial_{u^\alpha} + \zeta_i^\alpha \partial_{u_i^\alpha} + \zeta_{i_1 i_2}^\alpha \partial_{u_{i_1 i_2}^\alpha} + \cdots, \\
&\zeta_i^\alpha = D_i(\varkappa^\alpha) + \xi^j u_{ij}^\alpha, \qquad \zeta_{i_1 i_2}^\alpha = D_{i_1} D_{i_2}(\varkappa^\alpha) + \xi^j u_{j i_1 i_2}^\alpha, \qquad \varkappa^\alpha = \eta^\alpha - \xi^i u_i^\alpha,
\end{aligned} \tag{31}
$$

with the coordinates $\xi^i([z,u])$, $\eta^\alpha([z,u])$, $\zeta_i^\alpha([z,u]), \ldots$ being differential functions from the space $\mathcal{A}$. The set of all Lie–Bäcklund operators forms an infinite-dimensional Lie algebra $L_{\mathcal{B}}$, and an operator of the form $X_* = \xi^i D_i$ is the Lie–Bäcklund operator for the differential function $\xi^i([z,u])$. The set L_* of operators X_* forms an ideal in $L_{\mathcal{B}}$. This property allows introducing the notion of the equivalence of two Lie–Bäcklund operators $X_1, X_2 \in L_{\mathcal{B}}$ if $X_1 - X_2 \in L_*$ (written as $X_1 \sim X_2$). In particular, any Lie–Bäcklund operator $X \in L_{\mathcal{B}}$ is equivalent to operator (31) with $\xi^i = 0$,

$$
X \sim Y = X - \xi^i D_i = \varkappa^\alpha \partial_{u^\alpha}, \qquad \varkappa^\alpha \equiv \eta^\alpha - \xi^i u_i^\alpha. \tag{32}
$$

The operator Y is known as the *canonical representation of* X, and in notation (32), we imply the prolongation of the action of the operator on all higher-order derivatives according to formulae (31). It is essential that in the group of infinitesimal transformations $\mathcal{G}$ with operator (32) and the parameter a, only the dependent variables u^α change, while the independent variables z^i remain unchanged:

$$
u'^\alpha = u^\alpha + a\varkappa^\alpha + O(a^2), \qquad z'^i = z^i. \tag{33}
$$

This property allowed formulating the concept of symmetry groups of IDEs of form (29) as a local group of transformations $\mathcal{G}$ with operator (32), for which the form of the function F_σ remains unchanged for any value of the group parameter a. Differentiating the appropriate invariance condition (written for the function F_σ dependent on the transformed dependent variable u'^α) with respect to the group parameter a and passing to the limit $a \to 0$ yields the determining equations. In contrast to the basic DE case, these determining equations are generally nonlocal.

Using the canonical operator Y, we can write the invariance criterion for equation (29) with respect to an admitted group in an infinitesimal form:

$$
Y F_\sigma|_{[F_\sigma]} = 0, \qquad \sigma = 1+s, \ldots, q+s, \quad \text{where} \quad Y \equiv \int \mathrm{d}z\, \varkappa(z) \frac{\delta}{\delta u(z)}. \tag{34}
$$

Meaning a generalization of the action of the canonical group operator not only on differential functions but also on *functionals*, we use variational differentiation [17] in the definition of Y here. It can be verified by a direct calculation that the action of the operator Y on any differential function and its derivatives, $u, u_z, \ldots$ for example, gives the usual result: $Yu = \varkappa, Yu_z = D_z(\varkappa)$, etc. Hence, if $F_\sigma = 0$ is a usual DE, then formulae (34) result in local determining equations, and if $F_\sigma = 0$ has the form of a system of IDEs, then formulae (34) can be regarded as *nonlocal* determining equations, dependent on both local and nonlocal variables.

Treating local and nonlocal variables in the determining equations as independent variables allows separating these equations into local and nonlocal equations. The procedure for solving the local determining equations is performed in a standard way, using the Lie algorithm based on splitting of a system of overdetermined equations with respect to local variables and their derivatives. As a result, the expressions for the coordinates of the group operator are found, determining the so-called group of *intermediate* symmetries [17], which are used further in analysing the nonlocal determining equations. The procedure for solving the nonlocal determining equations is performed similarly, by substituting the coordinates of the found intermediate symmetry group operator in the nonlocal determining equations and splitting them using variational differentiation. Hence, constructing the symmetries for the nonlocal equations also becomes an algorithmic procedure [19]. These operations generalize the second step of the algorithm to the case where $\mathcal{RM}$ is an integral or integro-differential manifold.

Concluding this subsection we describe the operation of prolonging a Lie point symmetry group on the nonlocal variable defined, for example, by integral relation (30). To execute this operation we first write the group operator in the canonical form Y and then formally prolong it on the nonlocal variable J:

$$Y + \varkappa^J \partial_J \equiv \varkappa \partial_u + \varkappa^J \partial_J. \tag{35}$$

The integral relation between $\varkappa$ and $\varkappa^J$ is obtained by applying operator (35) to equation (30). Substituting there explicit expression for coordinate $\varkappa$ of the operator Y and calculating resulting integrals, we obtain the required coordinate $\varkappa^J$ of the prolonged operator,

$$\varkappa^J = \int \frac{\delta J(u)}{\delta u(z)} \varkappa(z)\, \mathrm{d}z \equiv \int \frac{\delta \mathcal{F}(u(z'))}{\delta u(z)} \varkappa(z)\, \mathrm{d}z\, \mathrm{d}z' = \int \mathcal{F}_u \varkappa(z)\, \mathrm{d}z. \tag{36}$$

For brevity, only the integration argument of a generator's coordinate is specified here.

3.3. Restricting the group $\mathcal{G}$ on the solution and constructing $\mathcal{RG}$

The group $\mathcal{G}$ found in step II and determined by operators (31) and (32) is generally wider than the RG of interest, which is related to a particular solution of a BVP. Hence, to obtain the RGS, we need step III, *restricting* the group $\mathcal{G}$ on a manifold determined by this particular solution. From the mathematical standpoint, this procedure consists in checking the vanishing conditions for a linear combination of coordinates $\varkappa_j^\alpha$ of a canonical operator equivalent to (31) on some particular BVP solution $U^\alpha(z)$,

$$\left\{ \sum_j A^j \varkappa_j^\alpha \equiv \sum_j A^j \left(\eta_j^\alpha - \xi_j^i u_i^\alpha \right) \right\} \Bigg|_{u^\alpha = U^\alpha(z)} = 0. \tag{37}$$

The form of the condition set by relation (37) is common for any solution of the BVP, but how the restriction procedure of a group is realized may differ in each partial case. In the general scheme (given at the beginning of the section), it is related to the dashed arrow connecting the 'initial object' (a perturbative solution of a particular BVP) to the object arising as a result of step III.

In calculating combination (37) on a particular solution $U^\alpha(z)$, the latter is transformed from a system of DEs for group invariants to algebraic relations. Note two consequences of step III. First, the restriction procedure results in a set of relations between A^j and thus 'links' the coordinates of various group operators X_j admitted by $\mathcal{RM}$ (5), (29). Second, it (partially or completely) eliminates an arbitrariness that can arise in the values of the coordinates ξ^i and η^α in the case of an infinite group $\mathcal{G}$.

As a rule, the procedure of restricting the group $\mathcal{G}$ reduces its dimension. After performing this procedure a general element (31) of a new group $\mathcal{RG}$ is represented by a linear combination of new generators R_i with coordinates $\hat{\xi}^i$ and $\hat{\eta}^\alpha$ and arbitrary constants B^j:

$$X \Rightarrow R = \sum_j B^j R_j, \qquad R_j = \hat{\xi}^i_j \partial_{x^i} + \hat{\eta}^\alpha_j \partial_{u^\alpha}. \tag{38}$$

The set of operators R_j, each containing the required solution of a problem in the invariant manifold, defines a group of transformations $\mathcal{RG}$, which we also call RG (by analogy with the RG for models with DEs).

3.4. Constructing an RG-invariant solution

The three steps described above completely form the regular algorithm for constructing the RGS, but to finish a final step is needed. Step IV uses the RGS operators to find analytic expressions for new, improved BVP solutions (compared with the input perturbative solution).

From the mathematical standpoint, realizing this step involves use of *RG-invariance* conditions set by a *joint* system of equations (5) and (29) and the vanishing conditions for a linear combination of the coordinates $\hat{\varkappa}^\alpha_j$ of the canonical operator equivalent to (38),

$$\sum_j R^j \hat{\varkappa}^\alpha_j \equiv \sum_j B^j \left(\hat{\eta}^\alpha_j - \hat{\xi}^i_j u^\alpha_i\right) = 0. \tag{39}$$

The need to use $\mathcal{RM}$ in constructing the BVP solution is shown in the scheme by the dashed arrow connecting these objects.

Specification of step IV concludes the description of regular algorithm of RGS construction for models with IDEs. We note that last the two steps are basically the same as for models with DEs. The following section contains two examples showing the ability of the upgraded RGS algorithm.

4. Constructing RGSs for integral models

4.1. The example with solution functionals of Hopf equation

Return now to a simple illustrative example, discussed in section 2, i.e., an initial problem for Hopf equation (17). We have shown that its solution can be constructed by use any of RG algebra generators (25). Suppose we are interested not in the whole solution but only in some its characteristic at a given point, for example, a value of its first derivative at $x = 0$, which can be formally introduced by a linear functional of u,

$$u_x(z, 0) \equiv u^0_x = -\int_{-\infty}^{+\infty} \mathrm{d}x\, \delta'(x) u(t, x). \tag{40}$$

The z dependence of u^0_x can be easily restored by prolongation of a linear combination of RG generators (25) on solution functional (40). We again use the last generator in (25) in its simplest form with $\epsilon\psi^4 = 1$. Write then this generator in the canonical form and calculate its prolongation using formulae (35) and (36). Restricting the RG operator obtained after this prolongation on the space of the group variables $\{z, \epsilon, u^0_x\}$, we obtain the RG generator for solution functional (40). For the case $U = -x$, one arrives at generator

$$R_4 = \partial_\epsilon - z\left(u^0_x\right)^2 \partial_{u^0_x}. \tag{41}$$

The initial condition $u_x^0(z=0) = -1$ is known. Hence, use of invariant of this generator, $J^0 = \epsilon z - 1/u_x^0 = 1$, restores the desired dependence $u_x^0 = -1/(1-\epsilon z)$, which is valid from the point $z = 0$ up to the singularity point $z_{\rm sing} = 1/\epsilon$. Note that this result is obtained *without constructing the BVP solution*, using only an appropriate RGS. At first glance, the example considered and the construction performed seems clumsy, and it is easier to proceed from trivial solution (27). But for more complex situations, an explicit form of the solution is often unknown, but it is possible to construct the RGS (see, e.g., [13]).

4.2. The example of the RGS for solution of plasma kinetic equation

Here, in contrast to the previous (differential model) example, we consider the case where integral relations form the problem nucleus. A vivid example is the model used in the plasma kinetic theory to describe the Coulomb explosion of submicron plasmas in the field of multiterrawatt femto-second laser pulses that leads to ion acceleration at multi-MeV energies [26, 27]. The mechanisms and characteristics of ions triggered by the interaction of a short laser pulse with plasma are currently interesting because of their possible applications to novel-neutron-source development, x-ray source, proton radiography and isotope production.

The macroscopic state of cluster particles is governed by distribution functions f (for cluster ions with the mass M and charge Ze) that depend on time t, coordinate x and velocity v of a particle (for simplicity, we consider the one-dimensional plane geometry). Evolution of the distribution functions is described by the solution of the Cauchy problem for Vlasov kinetic equation supplemented by Poisson equation for electric field E,

$$f_t + vf_x + (Ze/M)Ef_v = 0, \qquad E_x - 4\pi Ze\int \mathrm{d}v\, f = 0, \qquad f|_{t=0} = f_0(x, v). \tag{42}$$

Study of even such a simplified model analytically meets essential difficulties, but use of the RG algorithm allows obtaining solutions for various initial particle distribution functions and finding the density, mean velocity and energy spectrum of the particles. To construct the RGS, consider a set of local and nonlocal equations in (42) and the evident constraint $E_v = 0$ for $\mathcal{RM}$. The Lie group of point transformations admitted by this manifold is calculated as indicated in [18, 19] and consists of the six generators

$$\begin{aligned} &X_0 = \partial_t, \qquad X_1 = \partial_x, \qquad X_2 = t\partial_x + \partial_v, \qquad X_3 = x\partial_x + v\partial_v - f\partial_f + E\partial_E, \\ &X_4 = 2t\partial_t + x\partial_x - v\partial_v - 3f\partial_f - 2E\partial_E, \qquad X_5 = (t^2/2)\partial_x + t\partial_v + (M/Ze)\partial_E, \end{aligned} \tag{43}$$

describing time and space translations (X_0 and X_1), Galilean boosts (X_2), dilations (X_3 and X_4) and the generator X_5. Finite transformations defined by X_5 correspond to transition to a coordinate system moving linearly with constant acceleration with respect to the laboratory system. Two commuting generators in (43), namely, the generator of Galilean boosts and the generator of the transition to a uniformly accelerated frame, appear as the desired RGS generators,

$$R_5 = (t^2/2)\partial_x + t\partial_v + (M/Ze)\partial_E, \qquad R_6 = t\partial_x + \partial_v. \tag{44}$$

Successive application of finite transformations defined by these generators shifts the initial coordinates h and velocities ν to new values in the phase space,

$$R(t, h, \nu) = h + \nu t + (Ze/2M)E(h)t^2, \qquad U(t, h, \nu) = \nu + (Ze/M)E(h)t, \tag{45}$$

function $E(h)$ being defined by the initial conditions

$$E(h) = 4\pi Ze\int_0^h \mathrm{d}y\int_{-\infty}^{\infty}\mathrm{d}v\, f_0(y, v). \tag{46}$$

The distribution function is the invariant of RGS generators (44) and is defined by $f_0(h, \nu)$ for any specified values h and ν (i.e. for a given group of particles). The distribution function that solves (42) is obtained by summing over all groups of particles, i.e. by integrating over the initial velocities and coordinates of plasma particles,

$$f(t, x, v) = \int_{-\infty}^{\infty} \mathrm{d}\nu \int_{-\infty}^{\infty} \mathrm{d}h \; f_0(\nu, h)\delta(x - R(t, h, \nu))\delta(v - U(t, h, \nu)). \tag{47}$$

For 'cold' cluster particles, $f_0 \propto \delta(\nu)$, we need only one RG generator, R_5, to construct the BVP solution. The zeroth and first moments of the distribution function with respect to velocity v yield the density and mean velocity distributions for the cluster ions, which allows us to estimate the maximum energy of the accelerated ions, the ion energy spectrum and the relation between this spectrum and the initial ion density distribution [26–28]. A similar approach to the spherical geometry [27] shows that an inhomogeneity of the initial cluster density distribution leads to a solution singularity after a finite time interval, even for initially immovable ions.

5. Conclusion

We have presented examples showing how the new algorithm applies to integro-differential systems. We presented a simple methodological, illustrative example here, analysing a solution characteristic for the Hopf equation (also see [13]). Evidence for the efficiency of the RGS algorithm in treating problems that had not yet been solved by other approaches is provided by less trivial examples: calculating the expansion of plasma bunches and ion clusters and the acceleration of ions in plasma kinetic theory in the plane [20, 21] and spherical [22] geometry; calculating integral characteristics for this expansion without knowing the solution itself [21, 23]; calculating nonlinear dielectric permittivities in plasma for an arbitrary order of nonlinearity [24]; obtaining a reduced description of the light beam intensity behaviour on the beam axes [13, 14, 24]; establishing an interrelation between physical quantities in various representations related by some integral transformation [24]; obtaining an approximate analytic description of the nonlinear Langmuir oscillations in expanding plasma via an approximate RGS for the integro-differential Vlasov–Poisson model with a self-consistent electromagnetic field [25].

In formulating and discussing the analytic form of the results in our publications concerning the RGS algorithm, we emphasize the role of invariants of appropriate RG operators. The manifested common regularities were considered in [24, 13] and formulated as the so-called Φ-theorem, which is a generalization of the well-known Π-theorem. We considered the relation of representations of BVP solutions with the general form predicted by the concept of functional self-similarity [29, 30] and the well-known principles of group analysis.

The results described in this report testify the universality of the RGS method. Therefore, they allow us to look for a further expansion of the class of problems that allow us to use the RGS method and to new objects for which the use of RGS algorithm is not yet a standard procedure. We have in mind infinite systems of integro-differential equations, similar to systems for correlation functions in statistical physics and to systems of the equations for 'dressed' Green's functions (propagators and vertex functions) in QFT.

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